14.4/14.7 Tangent Planes & Max/Min

Remember from Math 124:

The equation of the tangent line to the curve $y = f(x)$ at x_0 is given by

$$y - f(x_0) = f'(x_0)(x - x_0)$$

Similarly, the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0) is given by

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Entry Tasks:

1. Find the equation for the tangent line to $f(x) = x^3$ at $x = 2$.

$f(2) = (2)^3 = 8 = \text{height}$

$f'(x) = 3x^2$

$f'(2) = 3(2)^2 = 12 = \text{slope of tangent}$

$\boxed{y - 8 = 12(x - 2)}$ TANGENT LINE

2. Find the equation for the tangent plane to $f(x, y) = x^2 + 3y^2 x - y^3$ at $(x, y) = (2, 1)$.

$f(2,1) = (2)^2 + 3(1)^2(2) - (1)^3 = 4 + 6 - 1 = 9 = \text{HEIGHT}$

$f_x = 2x + 3y^2 \Rightarrow f_x(2,1) = 4 + 3 = 7$
= slope of tangent
in x-direction

$f_y = 6yx - 3y^2 \Rightarrow f_y(2,1) = 12 - 3 = 9$
= slope of tangent
in y-direction

$\boxed{z - 9 = 7(x - 2) + 9(y - 1)}$ TANGENT PLANE

Visuals and Derivation of Tangent Plane
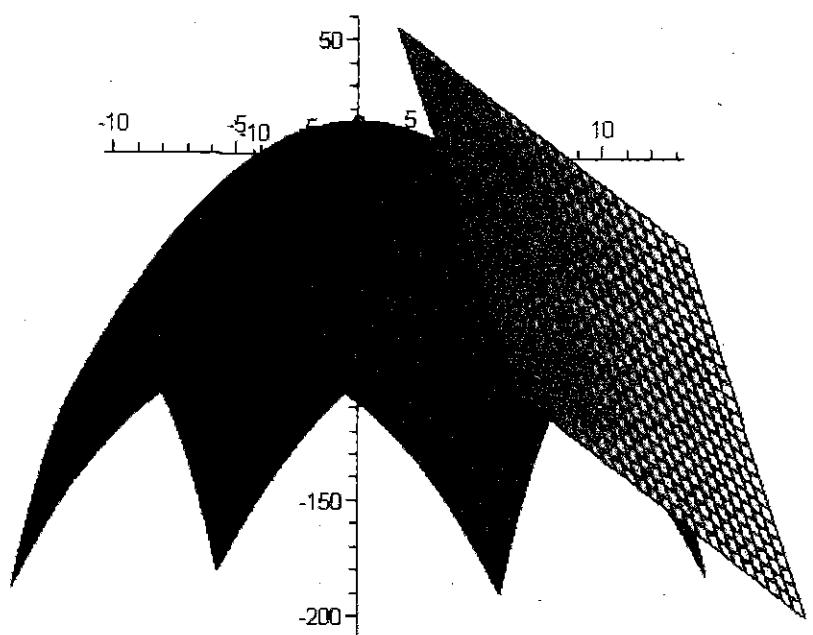
TANGENT PLANE

$$z - -50 = -14(x-7) - 8(y-4)$$

$z = f(x,y) = 15 - x^2 - y^2$ at $(7,4)$

Recall:

$f(7,4) = -50$

$f_x(7,4) = -14$

$f_y(7,4) = -8$

Derivation of Tangent Plane

$$\Box(x-\underline{\ \ }) + \Box(y-\underline{\ \ }) + \Box(z-\underline{\ \ }) = 0$$

The plane goes thru (7, 4, -50).
Now we need a normal vector.

Note:

$f_x(x,y) = -2x$

$f_x(7,4) = -14$
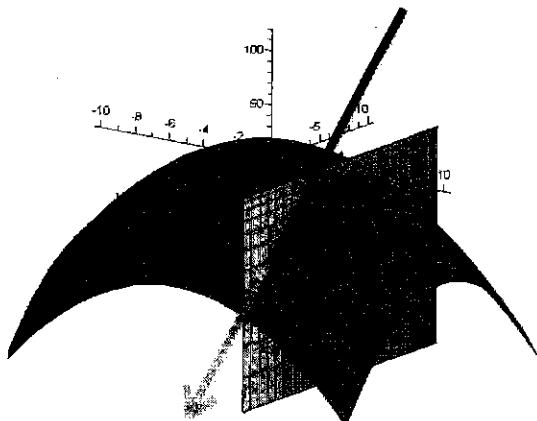
$f_y(x,y) = -2y$

$f_y(7,4) = -8$
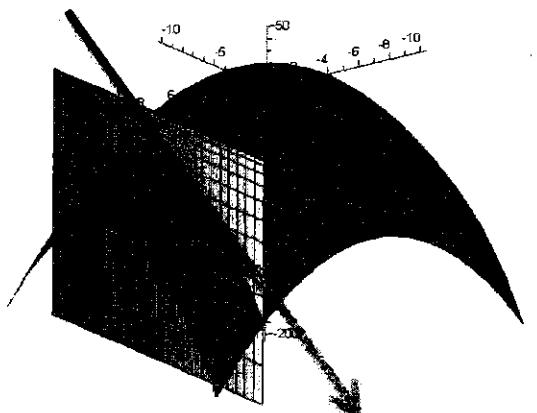
Thus, we can get two vectors that are parallel to the plane:

$$\langle 1,0,f_x(x_0,y_0)\rangle = \langle 1,0,-14\rangle$$
$$\langle 0,1,f_y(x_0,y_0)\rangle = \langle 0,1,-8\rangle$$

So a normal vector is given by

$$\langle 1,0,-14\rangle \times \langle 0,1,-8\rangle = \langle 14, 8, 1\rangle$$
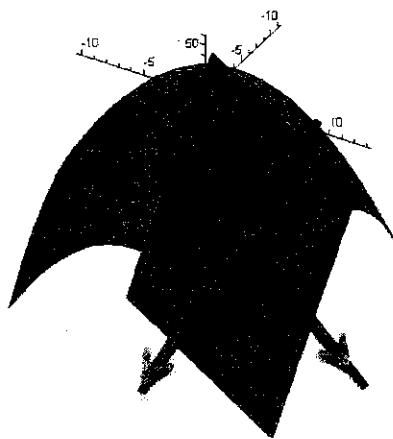
Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

Linear Approximation: "Near" the point (x_0, y_0) the tangent plane and surface have z-values that are close together.

Take $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$, add z_0 to both sides to get

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

We call the above function, $L(x, y)$, the linear approximation to $f(x, y)$ at (x_0, y_0).

Idea: $f(x, y) \approx L(x, y)$ for $(x, y) \approx (x_0, y_0)$

Example: Use the linear approximation to $f(x, y) = x^2 + 3y^2 x - y^3$ at $(x, y) = (2, 1)$ to estimate the value of $f(1.9, 1.05)$.

$$\underbrace{x^2 + 3y^2 x - y^3}_{f(x,y)} \approx \underbrace{9 + 7(x-2) + 9(y-1)}_{L(x,y)}$$

$$L(1.9, 1.05) = 9 + 7(1.9 - 2) + 9(1.05 - 1) = 9 - 0.7 + 0.45$$
$$= 8.75 \quad \longleftarrow \text{ linear approximation}$$

$$f(1.9, 1.05) = (1.9)^2 + 3(1.05)^2(1.9) - (1.05)^3 \approx 8.736625 \quad \longleftarrow \text{ACTUAL VALUE}$$

14.7 Local Max/Min

Consider the surface z = f(x,y).

Some Terminology:

A **local maximum** occurs at (a,b) if f(a,b) is larger than *all* values "near" it (top of a hill).

A **local minimum** occurs at (a,b) if f(a,b) is smaller than *all* values "near" it (bottom of a valley).

A **critical point** is a point (a,b) where **BOTH**

$$f_x(a,b) = 0 \quad \text{AND} \quad f_y(a,b) = 0$$

or where either partial doesn't exist.

If $f_x(a,b) = 0$ and $f_y(a,b) = 0$, and (a,b) is not a local max or min, then we call it a **saddle point**.

Example: Find the critical points of

$$f(x,y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$\boxed{1}\ f_x = 3y + 6x^2 + 9x \overset{?}{=} 0$

$\boxed{2}\ f_y = 3x - y \overset{?}{=} 0$

REMINDER OF HOW TO SOLVE A SYSTEM OF EQUATIONS.

$\boxed{\text{STEP 1}}$ Solve for one variable in one equation (whichever is easier)

$\boxed{2} \Rightarrow 3x = y$

$\boxed{\text{STEP 2}}$ Substitute into other equation.

$\boxed{2} \ \& \ \boxed{1} \Rightarrow 3(3x) + 6x^2 + 9x \overset{?}{=} 0$

$$6x^2 + 18x = 0$$

$\boxed{\text{STEP 3}}$ Solve!

$$6x(x+3) = 0$$

$x = 0 \quad$ or $\quad x+3 = 0$
$\qquad\qquad\qquad\qquad x = -3$

TWO ANSWERS

$x = 0 \Rightarrow y = 3(0) = 0 \Rightarrow (x,y) = (0,0)$

$x = -3 \Rightarrow y = 3(-3) = -9 \Rightarrow (x,y) = (-3,-9)$

CHECK!

Example: Find and classify all critical points for

$$f(x, y) = 3xy - \frac{1}{2}y^2 + 2x^3 + \frac{9}{2}x^2$$

$f_{xx} = 12x + 9$

$f_{yy} = -1$

$f_{xy} = 3 \quad = f_{yx}$

$D = f_{xx} f_{yy} - f_{xy} f_{yx} = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$

$\underbrace{\qquad}_{(f_{xy})^2}$

$D = (12x + 9)(-1) - (3)^2$

$(0,0) \Rightarrow D = (9)(-1) - 9 = -18 < 0$

$(-3,-9) \Rightarrow D = \underbrace{(-36 + 9)}_{=}\underbrace{(-1)}_{=} - 9 = 27 - 9 = 18 > 0$

LOCAL MAX $\quad \leftarrow \quad (x,y) = (-3,-9)$
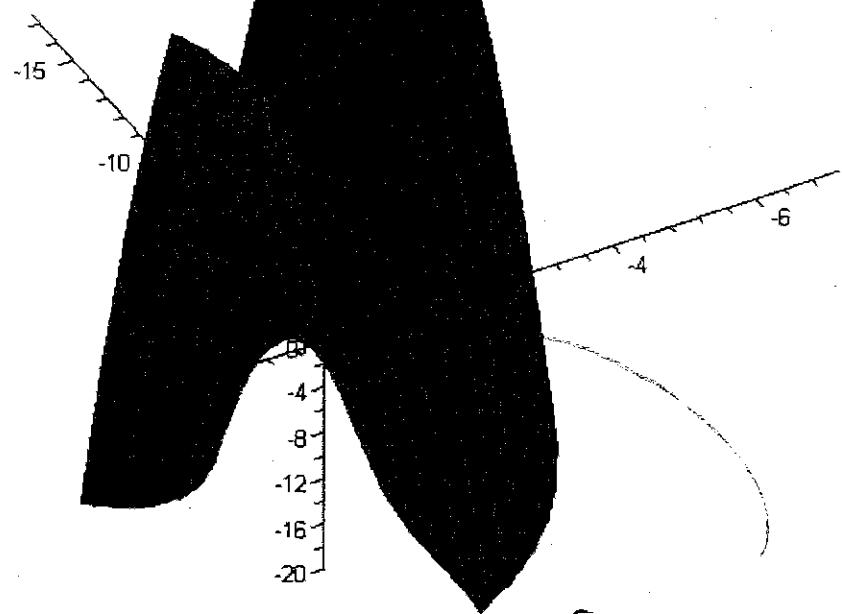
$(x,y) = (0,0)$
SADDLE POINT

ASIDE: LOCAL MAX <u>VALUE</u> = OUTPUT = $f(-3,-9) =$

$3(-3)(-9) - \frac{1}{2}(-9)^2 + 2(-3)^3 + \frac{9}{2}(-3)^2$

$= \boxed{27} = \text{LOCAL MAX}$

Second Derivative Test

Let (a,b) be a critical point.

Find all **second** partials at (a,b) and compute

$$D = f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

1. If **D > 0**, then the concavity is the same in all directions. So

   (a) If $f_{xx} > 0$, then it is concave up in all directions. **Local Minimum.** $\Rightarrow f_{yy} > 0$ As well

   (b) If $f_{xx} < 0$, then it is concave down in all directions. **Local Maximum.** $\Rightarrow f_{xx} < 0$ As well

2. If **D < 0**, then the concavity changes in some direction. **Saddle Point.**

3. If **D = 0**, the test is **inconclusive**.
   (need a contour map)

Quick Examples:

1. $f(x,y) = 15 - x^2 - y^2$,

Critical pt: $(0,0)$.

$f_{xx} = -2$, $f_{yy} = -2$, $f_{xy} = 0$

$D = (-2)(-2)-(0)^2 = 4$

$D > 0$, $f_{xx} < 0$, $f_{yy} < 0$

Local max!
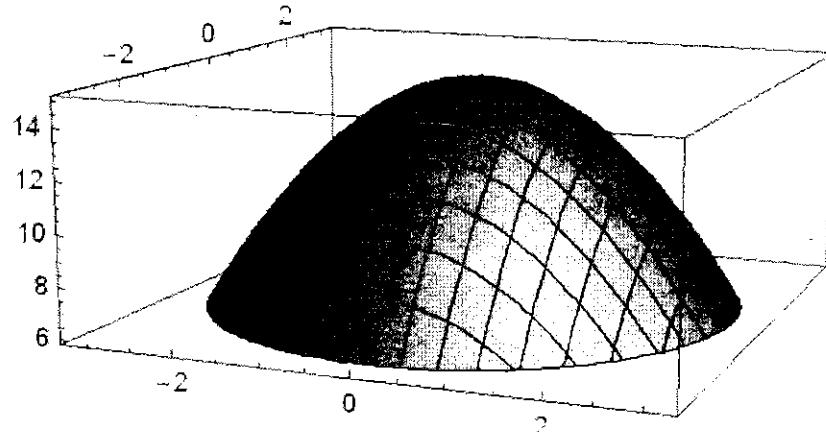
2. $f(x,y) = x^2 + y^2$,

Critical pt: $(0,0)$.

$f_{xx} = 2$, $f_{yy} = 2$, $f_{xy} = 0$,

$D = (2)(2)-(0)^2 = 4$

$D > 0$, $f_{xx} > 0$, $f_{yy} > 0$

Local min!
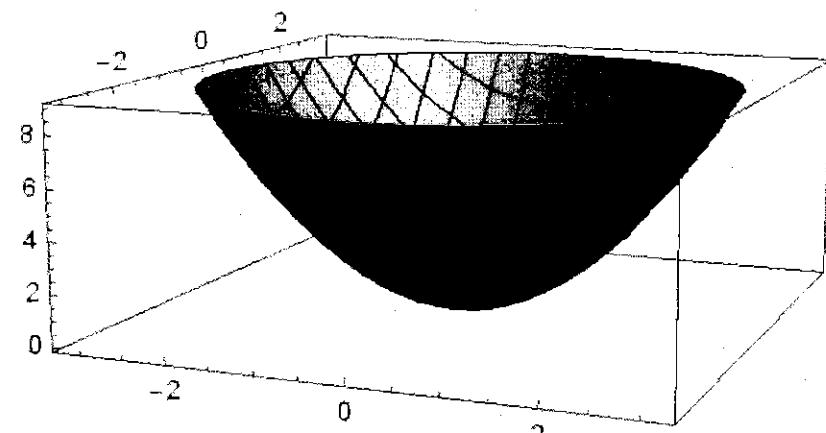
3. $f(x,y) = x^2 - y^2$

Critical pt: $(0,0)$.

$f_{xx} = 2$, $f_{yy} = -2$, $f_{xy} = 0$,

$D = (2)(-2)-(0)^2 = -4$

$D < 0$ (note also, $f_{xx} < 0$, $f_{yy} > 0$)

Saddle point!
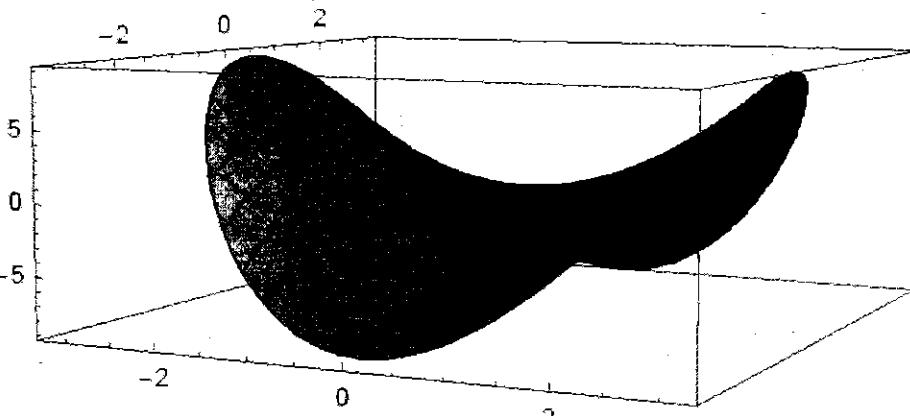
Examples from old exams:

1. Find and classify all critical points for

$$f(x,y) = x^2 + 4y - x^2y + 1$$

$\boxed{1}$ $\quad 2x - 2xy \overset{?}{=} 0$

$\boxed{2}$ $\quad 4 - x^2 \overset{?}{=} 0$

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$\boxed{1} \Rightarrow 2x(1-y) = 0 \Rightarrow x=0 \text{ or } y=1$

$\boxed{2} \Rightarrow x^2 = 4 \Rightarrow x=-2 \text{ or } x=2$

WHAT WORKS IN $\underline{\underline{BOTH}}$!!!

$x=0$ DOES $\underline{NOT}$ WORK IN BOTH.

$x=-2, y=1$ DOES WORK IN BOTH

$x=2, y=1$ DOES WORK IN BOTH

$\boxed{(-2,1) \quad (2,1)}$

$f_{xx} = 2-2y, \quad f_{yy} = 0, \quad f_{xy} = -2x$

$D = f_{xx}f_{yy} - (f_{xy})^2 = 0 - 4x^2$

At $(-2,1)$ D<0   SADDLE POINT

At $(2,1)$ D<0   SADDLE POINT

2. Find and classify all critical points for

$$f(x,y) = \frac{9}{x} + 3xy - y^2$$

$\boxed{1}$ $\quad f_x = -\frac{9}{x^2} + 3y \overset{?}{=} 0$

$\boxed{2}$ $\quad f_y = 3x - 2y \overset{?}{=} 0$

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$\boxed{2} \Rightarrow 3x = 2y \Rightarrow y = \frac{3}{2}x$

$\boxed{1} \text{ \& } \boxed{2} \Rightarrow -\frac{9}{x^2} + 3\left(\frac{3}{2}x\right) \overset{?}{=} 0$

$-\frac{9}{x^2} + \frac{9}{2}x = 0 \quad \left]\text{MULT. BY } x^2\right.$

$-9 + \frac{9}{2}x^3 = 0$

$\frac{9}{2}x^3 = 9$

$x^3 = 2$

$x = (2)^{1/3}$

$\Rightarrow y = \frac{3}{2}(2)^{1/3}$

$\boxed{\left(2^{1/3}, \frac{3}{2}(2)^{1/3}\right)}$

$f_{xx} = \frac{18}{x^3}, \quad f_{yy} = -2, \quad f_{xy} = 3$

$D = \frac{18}{x^3}(-2) - (3)^2$

FOR $x = 2^{1/3}$ D<0   SADDLE POINT

3. Find and classify all critical points for

$$f(x,y) = x^2y - 9y - xy^2 + y^3$$

① $f_x = 2xy - y^2 \overset{?}{=} 0$

② $f_y = x^2 - 9 - 2xy + 3y^2 \overset{?}{=} 0$

① $\Rightarrow y(2x - y) = 0$

$\Rightarrow y = 0$ OR $\underbrace{2x - y = 0}_{2x = y}$

TWO CASES

i) $y = 0 \Rightarrow$ ② $x^2 - 9 - 0 + 0 = 0$ $\Big\}$ $(-3, 0)$
$\Rightarrow x = \pm 3$ $(3, 0)$

ii) $y = 2x \Rightarrow$ ② $x^2 - 9 - 2x(2x) + 3(2x)^2 \overset{?}{=} 0$

$x^2 - 9 - 4x^2 + 3 \cdot 4x^2 \overset{?}{=} 0$

$-9 + 9x^2 = 0$

$9x^2 = 9$

$x^2 = 1$

$x = \pm 1$

$\Rightarrow y = 2x = \pm 2$

$\Big\}$ $(-1, -2)$
    $(1, 2)$

$f_{xx} = 2y, \quad f_{yy} = -2x + 6y$

$f_{xy} = 2x - 2y$

$(-3, 0) \Rightarrow f_{xx} = 0, f_{yy} = 6, f_{xy} = -6,$

$D = (0)(6) - (-6)^2 < 0$

SADDLE POINT

$(3, 0) \Rightarrow f_{xx} = 0, f_{yy} = -4, f_{xy} = 6$

$D = (0)(4) - (6)^2 < 0$

SADDLE POINT

$(-1, -2) \Rightarrow f_{xx} = -4, f_{yy} = -10$

$f_{xy} = 2$

$D = 40 - (2)^2 = 36 > 0$

LOCAL MAX

$(1, 2) \Rightarrow f_{xx} = 4, f_{yy} = 10$

$f_{xy} = -2$

$D = 40 - (-2)^2 = 36 > 0$

LOCAL MIN